STUDY MATERIALS

TOPIC: $\boxed{\text{LINE INTEGRALS}}$

MATHEMATICS HONS.

SEM- 4

<u>PAPER- C9 UNIT- 3</u>

Dr. Sangita Chakraborty

Associate Professor
Deptt. Of Mathematics
Kharagpur College

VECTOR INTEGRATION [Line Integration]

Ordinary Integrals: Let $\bar{R}(u) = R_1(u)\hat{i} + R_2(u)\hat{j} + R_3(u)\hat{k}$ be a vector function of u, where $R_1(u), R_2(u), R_3(u)$ are continuous in a specified interval $a \le u \le b$. Then

$$\int_{u=a}^{b} \bar{R}(u)\,du = \hat{i}\int_a^b R_1(u)\,du + \hat{j}\int_a^b R_2(u)\,du + \hat{k}\int_a^b R_3(u)\,du.$$

$$= \int_{u=a}^{b} \frac{d}{du}\left(\bar{S}(u)\right)du = \left[\bar{S}(u) + \bar{c}\right]_{u=a}^{b} = \bar{S}(b) - \bar{S}(a)$$

① **LINE INTEGRALS** :- Let $\bar{r}(u) = x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k}$, define a curve C be the position vector of (x, y, z), joining points $P_1\,(u=u_1)$ and $P_2\,(u=u_2)$.
Let $\bar{r}(u)$ have continuous derivative along C.
Let $\bar{A}(x,y,z) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ be a vector function of position and it be continuous along C.
Then the integral of the tangential component of $\bar{A}$ along C from P_1 to P_2 as a Line Integral:

$$\int_{P_1}^{P_2} \bar{A}\cdot d\bar{r} = \int_C \bar{A}\cdot d\bar{r} = \int_C A_1\,dx + A_2\,dy + A_3\,dz$$

If C be a simple closed curve, then the line integral:

$$\oint \bar{A}\cdot d\bar{r} = \oint A_1\,dx + A_2\,dy + A_3\,dz.$$

Work done $= \int_C \bar{F}\cdot d\bar{r}$, where $\bar{A} = \bar{F}(x,y,z)$, force acting on the particle moving along C.

Theorem: If $\bar{A} = \bar{\nabla}\phi$ everywhere in a region $R : \{a_1 \le x \le a_2,\ b_1 \le y \le b_2,\ c_1 \le z \le c_2\}$ where $\phi(x,y,z)$ is single-valued and has continuous derivatives in R, then

1. $\int_{P_1}^{P_2} \bar{A}\cdot d\bar{r}$ is independent of the path C in R joining P_1 & P_2

2. $\oint_C \bar{A}\cdot d\bar{r} = 0$ around any closed curve C in R.

$\bar{A} \longrightarrow$ conservative vector field iff $\bar{\nabla}\times\bar{A} = \bar{0}$ or, $\bar{A} = \bar{\nabla}\phi$
$\phi \rightarrow$ its scalar potential

For such case, $\bar{A}\cdot d\bar{r} = \bar{\nabla}\phi\cdot d\bar{r}$
$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$
an exact differential.

Problem: (a) If $\bar{F} = \bar{\nabla}\phi$ (ϕ is single-valued and having continuous partial derivatives), show that the work-done in moving a particle from one point $P_1(x_1, y_1, z_1)$ in this field to another point $P_2(x_2, y_2, z_2)$ is independent of the path joining two points

(b) Conversely, if $\int \bar{F}.d\bar{r}$ is independent of the path C joining two points, show that $\exists$ a function ϕ s.t. $\bar{F} = \bar{\nabla}\phi$.

(a) Work done $= \int_{P_1}^{P_2} \bar{F}.d\bar{r} = \int_{P_1}^{P_2} \bar{\nabla}\phi.d\bar{r} = \int_{P_1}^{P_2} \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

$= \int_{P_1}^{P_2} \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi(P_1)$

$= \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)$

$\therefore$ Work done depends on two points P_1 and P_2 only, and not on the path joining them.

This is true only if $\phi(x, y, z)$ is single-valued at all points P_1 and P_2.

(b) Let $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and given that $\int_C \bar{F}.d\bar{r}$ is independent of the path C joining any two points, say (x_1, y_1, z_1) and (x, y, z) respectively.

Then $\phi(x, y, z) = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \bar{F}.d\bar{r} = \int_{(x_1,y_1,z_1)}^{(x,y,z)} F_1\,dx + F_2\,dy + F_3\,dz$

is independent of the path joining (x_1, y_1, z_1) and (x, y, z)

Now, $\phi(x, y, z) = \int_{(x_1,y_1,z_1)}^{(x,y,z)} \bar{F}.\frac{d\bar{r}}{ds}\,ds$

Differentiating w.r.t. s, we get

$\frac{d\phi}{ds} = \bar{F}.\frac{d\bar{r}}{ds}$; But $\frac{d\phi}{ds} = \bar{\nabla}\phi \cdot \frac{d\bar{r}}{ds}$.

So that $(\bar{\nabla}\phi - \bar{F}) \cdot \frac{d\bar{r}}{ds} = 0$. Since this must hold irrespective of $\frac{d\bar{r}}{ds}$, we have $\bar{F} = \bar{\nabla}\phi$.

Prob: (a) If $\bar{F}$ is a conservative field, prove that curl $\bar{F} = \bar{\nabla} \times \bar{F} = \bar{0}$, i.e., $\bar{F}$ is irrotational.

(b) Conversely, if $\bar{\nabla} \times \bar{F} = \bar{0}$, Prove that $\bar{F}$ is conservative

(a) $\bar{F}$ is conservative means $\int_c \bar{F} \cdot d\bar{r}$ is independent of the path c joining two points, and hence by previous problem, $\bar{F} = \bar{\nabla}\phi$.

Thus curl $\bar{F} = \bar{\nabla} \times \bar{F} = \bar{\nabla} \times \bar{\nabla}\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$

$\therefore \bar{\nabla} \times \bar{F} = \hat{i}.0 + \hat{j}.0 + \hat{k}.0$

$\therefore \underline{\bar{\nabla} \times \bar{F} = \bar{0}} \Rightarrow \underline{\bar{F} \text{ is irrotational}}.$

(b) If $\bar{\nabla} \times \bar{F} = \bar{0}$, then $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \bar{0}$

$$\Rightarrow \left.\begin{aligned} \frac{\partial F_3}{\partial y} &= \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} &= \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} &= \frac{\partial F_1}{\partial y} \end{aligned}\right\}$$

We must show that $\bar{F} = \bar{\nabla}\phi$.

The workdone in moving a particle from (x_1, y_1, z_1) to (x, y, z) in the force field F is

$$\int_C F_1(x,y,z)\,dx + F_2(x,y,z)\,dy + F_3(x,y,z)\,dz$$

where C is the path joining (x_1, y_1, z_1) and (x, y, z). Let us choose as a particular path the straight line segments from (x_1, y_1, z_1) to (x, y_1, z_1) to (x, y, z_1) to (x, y, z) and call $\phi(x, y, z)$ is called the workdone along this particular path.

Then $\phi(x,y,z) = \int_{x_1}^{x} F_1(x, y_1, z_1)\,dx + \int_{y_1}^{y} F_2(x,y,z_1)\,dy + \int_{z_1}^{z} F_3(x,y,z)\,dz$

$\therefore \frac{\partial \phi}{\partial z} = F_3(x,y,z) \longrightarrow (1)$

$\frac{\partial \phi}{\partial y} = F_2(x,y,z_1) + \int_{z_1}^{z} \frac{\partial F_3}{\partial y}(x,y,z)\,dz = F_2(x,y,z_1) + \int_{z_1}^{z} \frac{\partial F_2}{\partial z}(x,y,z)\,dz$

$= F_2(x,y,z_1) + F_2(x,y,z)\Big|_{z=z_1}^{z}$

$= F_2(x,y,z_1) + F_2(x,y,z) - F_2(x,y,z_1)$

$\therefore \frac{\partial \phi}{\partial y} = F_2(x,y,z) \longrightarrow (2)$,

$\frac{\partial \phi}{\partial x} = F_1(x, y_1, z_1) + \int_{y_1}^{y} \frac{\partial F_2}{\partial x}(x,y,z_1)\,dy + \int_{z_1}^{z} \frac{\partial F_3}{\partial x}(x,y,z)\,dz$

$= F_1(x, y_1, z_1) + \int_{y_1}^{y} \frac{\partial F_1}{\partial y}(x,y,z_1)\,dy + \int_{z_1}^{z} \frac{\partial F_1}{\partial z}(x,y,z)\,dz$

i.e., $\dfrac{\partial\phi}{\partial x} = F_1(x, y_1, z_1) + F_1(x, y, z_1)\Big|_{y=y_1}^{y} + F_1(x,y,z)\Big|_{z=z_1}^{z}$

$= F_1(\cancel{x, y_1, z_1}) + F_1(\cancel{x, y, z_1}) - F_1(\cancel{x, y_1, z_1}) + F_1(x, y, z) - F_1(\cancel{x, y, z_1})$

$\therefore \dfrac{\partial\phi}{\partial x} = F_1(x, y, z) \longrightarrow (3)$

$\therefore$ From (1), (2) & (3), we get

$\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k} = \dfrac{\partial\phi}{\partial x}\hat{i} + \dfrac{\partial\phi}{\partial y}\hat{j} + \dfrac{\partial\phi}{\partial z}\hat{k} = \bar{\nabla}\phi.$

Thus, a necessary and sufficient condition that a field $\bar{F}$ be conservative is that curl $\bar{F} = \bar{\nabla}\times\bar{F} = \bar{0}$

Prob: Show that a necessary and sufficient condition that $F_1dx + F_2dy + F_3dz$ be an exact differential is that $\bar{\nabla}\times\bar{F} = \bar{0}$ where $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

Suppose, $F_1dx + F_2dy + F_3dz = d\phi = \dfrac{\partial\phi}{\partial x}dx + \dfrac{\partial\phi}{\partial y}dy + \dfrac{\partial\phi}{\partial z}dz$, an exact differential.

Then x, y, z are independent variables,

$F_1 = \dfrac{\partial\phi}{\partial x},\ F_2 = \dfrac{\partial\phi}{\partial y},\ F_3 = \dfrac{\partial\phi}{\partial z}.$

$\therefore \bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k} = \dfrac{\partial\phi}{\partial x}\hat{i} + \dfrac{\partial\phi}{\partial y}\hat{j} + \dfrac{\partial\phi}{\partial z}\hat{k} = \bar{\nabla}\phi$

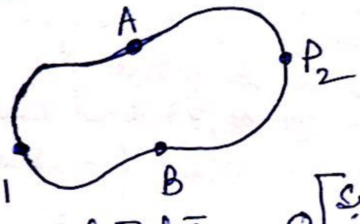
Thus $\bar{\nabla}\times\bar{F} = \bar{\nabla}\times\bar{\nabla}\phi = \bar{0}$

Conversely, if $\bar{\nabla}\times\bar{F} = \bar{0}$, then $\bar{F} = \bar{\nabla}\phi$, so that

$\bar{F}\cdot d\bar{r} = \bar{\nabla}\phi\cdot d\bar{r} = d\phi$, i.e., $F_1dx + F_2dy + F_3dz = d\phi$, an exact differential.

Prob: Prove that if $\int_{P_1}^{P_2}\bar{F}\cdot d\bar{r}$ is independent of the path joining any two points P_1 and P_2 in a given region, then $\oint\bar{F}\cdot d\bar{r} = 0$ for all closed paths in the region and conversely.

Let $P_1AP_2BP_1$ be a closed curve.



Then $\oint\bar{F}\cdot d\bar{r} = \displaystyle\int_{P_1AP_2BP_1}\bar{F}\cdot d\bar{r}$

$= \displaystyle\int_{P_1AP_2}\bar{F}\cdot d\bar{r} + \int_{P_2BP_1}\bar{F}\cdot d\bar{r} = \int_{P_1AP_2}\bar{F}\cdot d\bar{r} - \int_{P_1BP_2}\bar{F}\cdot d\bar{r} = 0$ [Since it is independent of the path]

$\therefore \oint\bar{F}\cdot d\bar{r} = 0$

Conversely, if $\oint\bar{F}\cdot d\bar{r} = 0$, then $\displaystyle\int_{P_1AP_2BP_1}\bar{F}\cdot d\bar{r} = \int_{P_1AP_2}\bar{F}\cdot d\bar{r} + \int_{P_2BP_1}\bar{F}\cdot d\bar{r}$

$= \displaystyle\int_{P_1AP_2}\bar{F}\cdot d\bar{r} - \int_{P_1BP_2}\bar{F}\cdot d\bar{r} = 0$

$\therefore \displaystyle\int_{P_1AP_2}\bar{F}\cdot d\bar{r} = \int_{P_1BP_2}\bar{F}\cdot d\bar{r}$

$\Rightarrow \displaystyle\int_{P_1}^{P_2}\bar{F}\cdot d\bar{r}$ is independent the path joining P_1 & P_2.

LINE INTEGRALS

Prob. 37 If $\bar{A} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$, evaluate $\int_C \bar{A}\cdot d\bar{r}$ along the following paths C:

(a) $x = 2t^2$, $y = t$, $z = t^3$ from $t=0$ to $t=1$,

(b) The straight lines from $(0,0,0)$ to $(0,0,1)$, then to $(0,1,1)$, and then to $(2,1,1)$

(c) the straight line joining $(0,0,0)$ and $(2,1,1)$.

(a) $\bar{A}\cdot d\bar{r} = (2y+3)\,dx + xz\,dy + (yz-x)\,dz$

$x = 2t^2, \quad dx = (4t)\,dt$

$y = t, \quad dy = dt$

$z = t^3, \quad dz = 3t^2\,dt$

$$\int_C \bar{A}\cdot d\bar{r} = \int_{t=0}^{1} (2t+3)\cdot 4t\,dt + 2t^2\cdot t^3\cdot dt + (t\cdot t^3 - 2t^2)\cdot 3t^2\,dt$$

$$= \int_{t=0}^{1} (8t^2 + 12t + 2t^5 + 3t^6 - 6t^4)\,dt = \left[\frac{8}{3}t^3 + \frac{12}{2}t^2 + \frac{2}{6}t^6 + \frac{3}{7}t^7 - \frac{6t^5}{5}\right]_{t=0}^{1}$$

$$= \frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5} = 9 + \frac{3}{7} - \frac{6}{5} = \frac{315+15-42}{35} = \frac{288}{35}.$$

(b) Straight line C_1: from $(0,0,0)$ to $(0,0,1)$,

here $x=0$, $y=0$, $z = 0$ to 1.

$\therefore dx = 0, dy = 0.$

$$\therefore \int_{C_1} \bar{A}\cdot d\bar{r} = \int_{z=0}^{1} (yz-x)\,dz = \int_{z=0}^{1} (0.z-0)\,dz = 0.$$

Straight line C_2: from $(0,0,1)$ to $(0,1,1)$.

$x=0$, $y = 0$ to 1, $z=1$; $\therefore dx = 0$, $dz = 0$.

$$\int_{C_2} \bar{A}\cdot d\bar{r} = \int_{y=0}^{1} xz\,dy = \int_0^1 0.1\,dy = 0.$$

Straight line C_3: from $(0,1,1)$ to $(2,1,1)$.

$x = 0$ to 2, $y=1$, $z=1$, $\therefore dy=0, dz=0$.

$$\int_{C_3} \bar{A}\cdot d\bar{r} = \int_{x=0}^{2} (2y+3)\,dx = \int_{x=0}^{2} (2.1+3)\,dx = \left[5x\right]_0^2 = 10.$$

$$\therefore \int_C \bar{A}\cdot d\bar{r} = \int_{C_1} \bar{A}\cdot d\bar{r} + \int_{C_2} \bar{A}\cdot d\bar{r} + \int_{C_3} \bar{A}\cdot d\bar{r} = 0 + 0 + 10 = 10$$

(c) Straight line joining $(0,0,0)$ to $(2,1,1)$ is given by

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t\text{ (say)} \Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{1} = t$$

$\therefore x = 2t, \ y = t, \ z = t. \quad dx = 2\,dt, \ dy = dt, \ dz = dt$

$$\int_C \bar{A}\cdot d\bar{r} = \int_{t=0} (2t+3)\cdot 2\,dt + 2t\cdot t\,dt + (t\cdot t - 2t)\,dt = \int_{t=0}^{1} (3t^2 + 2t + 6)\,dt$$

$= 1 + 1 + 6 + 8$

Prob. (39). If $\bar{F} = (2x+y)\hat{i} + (3y-x)\hat{j}$, evaluate $\int \bar{F}\cdot d\bar{r}$ where C is the curve in the xy plane consisting of the straight lines from $(0,0)$ to $(2,0)$ and then to $(3,2)$.

Solution: $F\cdot d\bar{r} = (2x+y)\,dx + (3y-x)\,dy$

The straightline C_1: from $(0,0)$ to $(2,0)$, $x = 0$ to 2, $y = 0$.

$\therefore dy = 0$.

$$\therefore \int_{C_1} \bar{F}\cdot d\bar{r} = \int_{x=0}^{2} (2x+y)\,dx = \int_0^2 2x\,dx = x^2\Big]_0^2 = 4.$$

The straightline C_2: from $(2,0)$ to $(3,2)$.

$$\therefore \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = t \Rightarrow \frac{x-2}{3-2} = \frac{y-0}{2-0} = t$$

t varies from 0 to 1. $\Rightarrow x = t+2,\ y = 2t$.

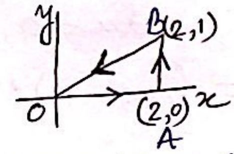
$dx = dt,\ dy = 2dt$.

$$\therefore \int_{C_2} \bar{F}\cdot d\bar{r} = \int_{t=0}^{1} \{2(t+2)+2t\}\,dt + (3\cdot 2t - t - 2)\cdot 2\,dt$$

$$= \int_{t=0}^{1} 14t\,dt = 7t^2\Big]_{t=0}^{1} = 7$$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int_{C_1} \bar{F}\cdot d\bar{r} + \int_{C_2} \bar{F}\cdot d\bar{r} = 4+7 = 11.$$

P. 43. If $F = (2x+y^2)\hat{i} + (3y-4x)\hat{j}$, evaluate $\oint_C F\cdot d\bar{r}$ around the triangle C:



$F\cdot d\bar{r} = (2x+y^2)\,dx + (3y-4x)\,dy.$

Along the straight line $\overrightarrow{OA}$: $x = 0$ to 2, $y = 0$, $\therefore dy = 0$.

$$\int_{\overrightarrow{OA}} F\cdot d\bar{r} = \int_{x=0}^{2} (2x+y^2)\,dx = \int_0^2 2x\,dx = \left[x^2\right]_0^2 = 4.$$

Along the Straight line $\overrightarrow{AB}$: $x = 2$, $y = 0$ to 1, $dx = 0$.

$$\int_{\overrightarrow{AB}} F\cdot d\bar{r} = \int_{y=0}^{1} (3y-4x)\,dy = \int_0^1 (3y-4\cdot 2)\,dy = \left[\frac{3y^2}{2} - 8y\right]_{y=0}^{1} = \left(3\cdot\frac{1}{2} - 8\right)$$

$$= \frac{3}{2} - 8 = \frac{-13}{2}$$

Along the Straight line $\overrightarrow{BO}$: $\frac{x-2}{0-2} = \frac{y-1}{0-1} = t$

$\therefore x = 2-2t,\ y = 1-t,\ dx = -2dt,\ dy = -dt,\ t = 0$ to 1

$$\int_{\overrightarrow{BO}} \bar{F}\cdot d\bar{r} = \int_{t=0}^{1} \{2(2-2t)+(1-t)^2\}(-2)\,dt + \{3(1-t) - 4(2-2t)\}\cdot(-dt)$$

$$= \int_{t=0}^{1} \{(t^2-6t+5)\times(-2) + (5t-5)\times(-1)\}\,dt = \int_{t=0}^{1} (-2t^2+7t-5)\,dt = -\frac{2}{3} + \frac{7}{2} - 5$$

$$= \frac{-4+21-30}{6}$$

$$= \frac{-13}{6}$$

$$\therefore \oint_C F\cdot d\bar{r} = \int_{OA} + \int_{AB} + \int_{BO} = 4 - \frac{13}{2} - \frac{13}{6} = \frac{24-39-13}{6} = -\frac{28}{6} = \frac{-14}{3}$$

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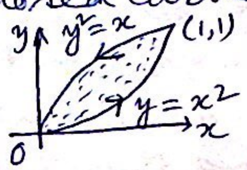
44. Evaluate $\oint_C \bar{A}\cdot d\bar{r}$ around the closed curve C if $\bar{A} = (x-y)\hat{i} + (x+y)\hat{j}$



Soln: Along $y = x^2$, $dy = 2x\,dx$.

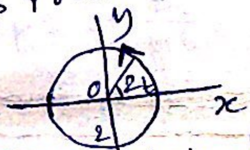
Now, $\bar{A}\cdot d\bar{r} = (x-y)\,dx + (x+y)\,dy$.

$$\therefore \int_{(y=x^2)} \bar{A}\cdot d\bar{r} = \int_{x=0}^{1} (x-x^2)\,dx + (x+x^2)\,2x\,dx = \int_0^1 (2x^3+x^2+x)\,dx = \frac{2}{4}+\frac{1}{3}+\frac{1}{2} = 1+\frac{1}{3} = 4/3.$$

$$\int_{y^2=x} \bar{A}\cdot d\bar{r} = \int_{y=1}^{0} (y^2-y)\,2y\,dy + (y^2+y)\,dy = \int_{y=1}^{0} (2y^3-y^2+y)\,dy = -\frac{2}{4}+\frac{1}{3}-\frac{1}{2} = -1+\frac{1}{3} = -2/3.$$

$$\therefore \oint_C \bar{A}\cdot d\bar{r} = \frac{4}{3}-\frac{2}{3} = \frac{2}{3}.\ \text{(Ans.)}$$

45. If $\bar{A} = (y-2x)\hat{i} + (3x+2y)\hat{j}$, compute the circulation of $\bar{A}$ about a circle C in the xy plane with centre at the origin and radius 2, if C is traversed in the +ve direction.



$\bar{A}\cdot d\bar{r} = (y-2x)\,dx + (3x+2y)\,dy$

Circle C: $x^2+y^2 = 2^2$, $x = 2\cos t$, $y = 2\sin t$, t is a parameter

t runs for 0 to 2π.

$$\therefore \int_C \bar{A}\cdot d\bar{r} = \int_{t=0}^{2\pi} (2\sin t - 4\cos t)(-2\sin t)\,dt + (3\cdot 2\cos t + 2\cdot 2\sin t)(2\cos t)\,dt = \int_0^{2\pi} (-4\sin^2 t + 16\sin t\cos t + 12\cos^2 t)\,dt$$

$$= \int_{t=0}^{2\pi} (-4\sin^2 t + 8\sin t\cos t + 12\cos^2 t + 8\sin t\cos t)\,dt = \int_0^{2\pi} (4+8\cos 2t+8\sin 2t)\,dt$$

$$= \int_0^{2\pi} [-2(1-\cos 2t) + 8\sin 2t + 6(1+\cos 2t)]\,dt$$

$$= \int_0^{2\pi} 4\,dt + 8\int_0^{2\pi} \cos 2t\,dt + 8\int_0^{2\pi} \sin 2t\,dt = 4\times 2\pi + 8\left[\frac{\sin 2t}{2} - \frac{\cos 2t}{2}\right]_{t=0}^{2\pi}$$

$$= 8\pi + 4[1-1] = 8\pi \ \text{Ans.}$$

47. Prove that $\bar{F} = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2+2)\hat{k}$ is a conservative force field.

(i) Find the scalar potential for $\bar{F}$

(ii) Find the work done in moving an object in this field from $(0,1,-1)$ to $(\pi/2,-1,2)$.

Ans: A necessary and sufficient condition that a force $\bar{F}$ will be conservative is that curl $\bar{F} = \bar{\nabla}\times\bar{F} = \bar{0}$

Now $\bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2\cos x+z^3) & (2y\sin x-4) & (3xz^2+2) \end{vmatrix} = 0\hat{i}+0\hat{j}+0\hat{k} = \bar{0}$

$\therefore$ F is a conservative force field.

(i) $\therefore$ $F = \bar{\nabla}\phi$, ϕ is a scalar potential for F

$\therefore$ $\frac{\partial\phi}{\partial x} = y^2\cos x + z^3 \Rightarrow \phi(x,y,z) = y^2\sin x + xz^3 + f(y,z)$

$\frac{\partial\phi}{\partial y} = 2y\sin x - 4 \Rightarrow \phi(x,y,z) = y^2\sin x - 4y + g(x,z)$

$\frac{\partial\phi}{\partial z} = 3xz^2 + 2 \Rightarrow \phi(x,y,z) = xz^3 + 2z + h(x,y)$

Let us choose $f(y,z) = -4y+2z$, $g(x,z) = xz^3+2z$, & $h(x,y) = y^2\sin x - 4y$.

Then $\phi(x,y,z) = y^2\sin x + xz^3 - 4y + 2z$ + Constant.

(ii) The work done $= \int_{P_1}^{P_2} \bar{F}\cdot d\bar{r} = \int_{P_1}^{P_2} \bar{\nabla}\phi\cdot d\bar{r} = \int_{P_1}^{P_2} d\phi = \phi(P_2)-\phi(P_1)$

$= \phi(\pi/2,-1,2) - \phi(0,1,-1)$

$= 1 + 4\pi + 4 + 4 + \cancel{0} + 4 + 2 - \cancel{0} = \underline{15+4\pi}$ Ans.

53. Show that $(2x\cos y + z\sin y)dx + (xz\cos y - x^2\sin y)dy + x\sin y\,dz$ is an exact differential. Hence solve the corresponding D.E.

Ans: Let $F = (2x\cos y + z\sin y)\hat{i} + (xz\cos y - x^2\sin y)\hat{j} + x\sin y\hat{k}$

A necessary and sufficient condition that $\bar{F}\cdot d\bar{r}$ is an exact differential $= d\phi$, if $\bar{F} = \bar{\nabla}\phi$ or curl $\bar{F} = \bar{0}$.

Now, we see that curl $\bar{F} = \bar{0}$, Hence it is exact differential

Now, Solve an exact differential eqn

Ans: $x^2\cos y + xz\sin y$ = Constant.

Spiegel / Line Integration

40. Find the workdone in moving a particle in the force field $\bar{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along

ⓐ the straight line from $(0,0,0)$ to $(2,1,3)$.

ⓑ the space curve $x = 2t^2$, $y = t$, $z = 4t^2 - t$ from $t=0$ to $t=1$.

ⓒ the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x=0$ to $x=2$.

Solution:

ⓐ The parametric equations of the straight line from $(0,0,0)$ to $(2,1,3)$ is given by:

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t \text{ (say); or, } x = 2t,\ y = t,\ z = 3t$$

where $t: 0 \to 1$.

Now workdone $= \int F \cdot d\bar{r} = \int 3x^2 dx + (2xz - y)dy + z\,dz$

$$= \int_{t=0}^{1} \left[3\cdot(2t)^2(2dt) + (2\cdot 2t\cdot 3t - t)\,dt + 3t\,(3dt)\right]$$

$$= \int_{t=0}^{1} \left[24t^2 + 12t^2 - t + 9t\right]dt = \int_{t=0}^{1} (36t^2 + 8t)\,dt = \left[12t^3 + 4t^2\right]_{t=0}^{1}$$

$= 16.$

$\therefore$ workdone $= \int \bar{F}\cdot d\bar{r} = 16$. (Ans.)

ⓑ
$$\int \bar{F}\cdot d\bar{r} = \int_{t=0}^{1} \left[3\cdot(2t^2)^2 d(2t^2) + \{2\cdot 2t^2\cdot(4t^2 - t) - t\}dt + (4t^2 - t)\,d(4t^2 - t)\right]$$

$$= \int_{t=0}^{1} \left[48t^5 + 16t^4 - 4t^3 - t + (4t^2 - t)(8t - 1)\right]dt$$

$$= \int_{t=0}^{1} \left[48t^5 + 16t^4 + 28t^3 - 12t^2\right]dt$$

$$= \left[8t^6 + \frac{16}{5}t^5 + 7t^4 - 4t^3\right]_{t=0}^{1}$$

$$= 8 + \frac{16}{5} + 7 - 4 = 11 + \frac{16}{5} = \frac{71}{5}.$$

$\therefore$ Workdone $= \frac{71}{5}$ (Ans)

ⓒ The given curve: $x^2 = 4y$, $3x^3 = 8z$; or, $y = x^2/4$, $z = \frac{3}{8}x^3$.

Workdone $= \int \bar{F}\cdot d\bar{r} = \int 3x^2 dx + (2xz - y)\,dy + z\,dz$

$$= \int_{x=0}^{2} 3x^2 dx + \left(2x\cdot\frac{3}{8}x^3 - \frac{x^2}{4}\right) d\left(\frac{x^2}{4}\right) + \frac{3}{8}x^3 d\left(\frac{3}{8}x^3\right)$$

$$= \int_{x=0}^{2} \left[3x^2 + \left(\frac{3}{4}x^4 - \frac{x^2}{4}\right)\cdot\frac{x}{2} + \frac{27}{64}x^5\right]dx$$

$$= \int_{x=0}^{2} \left[3x^2 + \left(\frac{3}{8} + \frac{27}{64}\right)x^5 - \frac{x^3}{8}\right]dx = \left[x^3 + \frac{1}{6}\left(\frac{3}{8} + \frac{27}{64}\right)x^6 - \frac{x^4}{32}\right]_{x=0}^{2}$$

$$= 8 + \frac{51}{\cancel{64}}\cdot\frac{\cancel{2^6}}{6} - \frac{2^4}{32} = 8 + \frac{17}{2} - \frac{1}{2} = 16. \text{ (Ans.)}$$

Ghosh & Maity Book [C]

② Show that $\bar{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field. Find the scalar potential. Find also the workdone in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$.

1st part: The condition for $\bar{F}$ to be a conservative force field is that $\bar{\nabla}\times\bar{F} = \bar{0}$.

Now
$$\bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix} = \hat{i}\left\{\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(x^2)\right\} + \hat{j}\left\{\frac{\partial}{\partial z}(2xy+z^3) - \frac{\partial}{\partial x}(3xz^2)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy+z^3)\right\}$$
$$= 0\hat{i} + \hat{j}(3z^2 - 3z^2) + \hat{k}(2x - 2x)$$
$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \bar{0}$$

$\bar{F}$ is conservative. (proved)

2nd part: Since $\bar{F}$ is conservative, there exists a scalar potential ϕ s.t. $\bar{F} = \bar{\nabla}\phi$

i.e., $(2xy+z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$.

This give:
$$\frac{\partial\phi}{\partial x} = 2xy + z^3 \longrightarrow ①$$
$$\frac{\partial\phi}{\partial y} = x^2 \longrightarrow ②$$
$$\frac{\partial\phi}{\partial z} = 3xz^2 \longrightarrow ③$$

Integrating ①, ②, ③ partially w.r.t. x, y, z respectively, we obtain;
$$\left.\begin{aligned} \phi &= x^2y + xz^3 + f(y,z) \\ \phi &= x^2y + g(x,z) \\ \phi &= xz^3 + h(x,y) \end{aligned}\right\}$$
where $f(y,z)$, $g(x,z)$ and $h(x,y)$ are arbitrary functions.

Let us choose $f(y,z) = c_1$, $g(x,z) = xz^3 + c_2$, $h(x,y) = x^2y + c_3$

Then $\phi = x^2y + xz^3 + c$, where $c = c_1 + c_2 + c_3$, an arbitrary constant.

3rd part: Workdone $= \int \bar{F}\cdot d\bar{r} = \int \bar{\nabla}\phi \cdot d\bar{r}$
$$= \int\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$
$$= \int \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \int_{(1,-2,1)}^{(3,1,4)} d\phi$$
$$= [\phi]_{(1,-2,1)}^{(3,1,4)} = [x^2y + xz^3 + c]_{(1,-2,1)}^{(3,1,4)}$$
$$= \underline{202}.$$

④ Show that $(z^2-2xy)dx - x^2dy + 2xz\,dz$ is an exact differential. Find a function $\phi(x,y,z)$ such that $d\phi =$ the given differential.

The necessary and sufficient condition that $F_1dx + F_2dy + F_3dz$ be an exact differential is that $\bar{\nabla}\times\bar{F} = \bar{0}$ where $\bar{F} = (F_1, F_2, F_3)$.

$= (z^2-2xy, -x^2, 2xz)$, (given).

$$\therefore \bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (z^2-2xy) & (-x^2) & 2xz \end{vmatrix} = \hat{i}\left\{\frac{\partial}{\partial y}(2xz)-\frac{\partial}{\partial z}(-x^2)\right\}+\hat{j}\left\{\frac{\partial}{\partial z}(z^2-2xy)-\frac{\partial}{\partial x}(2xz)\right\}$$
$$+\hat{k}\left\{\frac{\partial}{\partial x}(-x^2)-\frac{\partial}{\partial y}(z^2-2xy)\right\}.$$
$$= \hat{i}\,0 + \hat{j}(2z-2z)+\hat{k}(-2x+2x)$$
$$= \bar{0}$$

$\therefore$ The given differential is an exact differential.

Let $d\phi = (z^2-2xy)dx - x^2dy + 2xz\,dz$

or, $\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z} = (z^2-2xy)dx - x^2dy + 2xz\,dz.$

Since x, y and z are independent variables,

$$\frac{\partial\phi}{\partial x} = z^2-2xy,\quad \frac{\partial\phi}{\partial y} = -x^2,\quad \frac{\partial\phi}{\partial z} = 2xz$$

Integrating partially w.r.t. x, y, z respectively,

$\phi = xz^2 - x^2y + f(y,z)$, $\phi = -x^2y + g(x,z)$, $\phi = xz^2 + h(x,y)$

where $f(y,z)$, $g(x,z)$ and $h(x,y)$ are arbitrary functions. Let us choose $f(y,z) = c_1$, $g(x,z) = xz^2+c_2$, $h(x,y) = -x^2y + c_3$.

Then $\underline{\phi(x,y,z) = xz^2 - x^2y + c}$, where $c = c_1+c_2+c_3$ an arbitrary constant

⑥ Evaluate $\int yz\,dx + (xz+1)dy + xy\,dz$ along any path Γ joining the points $(1,0,0)$ and $(2,1,4)$.

First of all we have to check whether $yz\,dx + (xz+1)dy + xy\,dz$ is an exact differential

Let $\bar{F} = (yz, xz+1, xy)$. Then $\bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz+1 & xy \end{vmatrix} = (x-x)\hat{i}+(y-y)\hat{j}+(z-z)\hat{k} = \bar{0}$

$\therefore$ It is an exact differential, so there exists a scalar function ϕ such that

$yz\,dx + (xz+1)dy + xy\,dz = d\phi$ (Exact differential)

or, $\int yz\,dx + (xz+1)dy + xy\,dz = \int d\phi = \phi + \text{Constant}$,

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Now we have:

$$\frac{\partial\phi}{\partial x} = yz, \quad \frac{\partial\phi}{\partial y} = xz+1, \quad \frac{\partial\phi}{\partial z} = xy.$$

Integrating w.r.t. x, y, z partially respectively,

$$\phi = xyz + f(y,z), \quad \phi = xyz + y + g(x,z), \quad \phi = xyz + h(x,y).$$

Let us choose $f(y,z) = y + c_1$, $g(x,z) = c_2$, $h(x,y) = y + c_3$

Then $\phi(x,y,z) = xyz + y + c$, where $c_1 + c_2 + c_3 = c$.

$$\therefore \int_\Gamma yz\,dx + (xz+1)\,dy + xy\,dz = \Big[\phi(x,y,z)\Big]_{(1,0,0)}^{(2,1,4)}$$

$$= \Big[xyz + y + c\Big]_{(1,0,0)}^{(2,1,4)}$$

$$= 8 + 1 - 0 = \underline{9} \text{ (Ans).}$$

Spiegel

(38) If $\bar{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$, evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve C in the xy plane, $y = x^3$ from the point $(1,1)$ to $(2,8)$.

$y = x^3$, then $dy = 3x^2\,dx$.

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (5xy - 6x^2)\,dx + (2y - 4x)\,dy.$$

$$= \int_{x=1}^{2} \left[(5x.x^3 - 6x^2) + (2x^3 - 4x)3x^2\right]dx$$

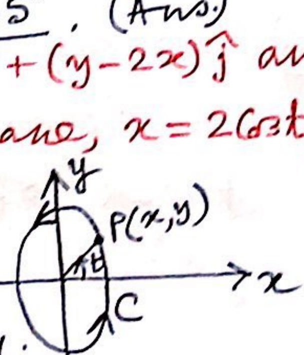
$$= \int_{x=1}^{2} (5x^4 - 6x^2 + 6x^5 - 12x^3)\,dx$$

$$= \left[x^5 - 2x^3 + x^6 - 3x^4\right]_{x=1}^{2}$$

$$= (32 - 16 + 64 - 48) - (1 - 2 + 1 - 3)$$

$$= 32 + 3 = \underline{35}. \text{ (Ans.)}$$

(41) Evaluate $\oint_C \bar{F}\cdot d\bar{r}$, where $\bar{F} = (x - 3y)\hat{i} + (y - 2x)\hat{j}$ and C is the closed curve in the xy plane, $x = 2\cos t$, $y = 3\sin t$ from $t = 0$ to $t = 2\pi$.



Let C be a closed curve which is traversed in the (+ve) counterclockwise direction.

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_C (x-3y)\,dx + (y-2x)\,dy.$$

$$= \int_{t=0}^{2\pi} \left[(2\cos t - 9\sin t)\,d(2\cos t) + (3\sin t - 4\cos t)\,d(3\sin t)\right]$$

$$= \int_{t=0}^{2\pi} \left[-2\sin 2t + 18\sin^2 t + \frac{9}{2}\sin 2t - 12\cos^2 t\right]dt = \int_{t=0}^{2\pi}\left[\frac{5}{2}\sin 2t - 12\cos 2t + 3(1 - \cos 2t)\right]dt$$

$$= \left[-\frac{5}{4}\cos 2t - 6\sin 2t + 3t - \frac{3}{2}\sin 2t\right]_{t=0}^{2\pi} = \underline{6\pi}. \text{ (Ans.)}$$

Spiegel (53)
Maity & Ghosh (10)

Show that $(2x\cos y + z\sin y)dx + (xz\cos y - x^2\sin y)dy + x\sin y\, dz$ is an exact differential. Hence solve the corresponding differential equation.

The necessary and sufficient condition that $F_1 dx + F_2 dy + F_3 dz$ an exact differential is $\bar{\nabla}\times\bar{F} = \bar{0}$. Here $\bar{F} = (2x\cos y + z\sin y)\hat{i} + (xz\cos y - x^2\sin y)\hat{j} + x\sin y\,\hat{k}$

Now,
$$\bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (2x\cos y + z\sin y) & (xz\cos y - x^2\sin y) & (x\sin y) \end{vmatrix}$$
$$= \hat{i}(x\cos y - x\cos y) + \hat{j}(\sin y - \sin y) + \hat{k}(z\cos y - 2x\sin y + 2x\sin y - z\cos y)$$
$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \bar{0}$$

$\therefore$ The given is an exact differential.

So, there exists a scalar function $\phi(x,y,z)$ s.t. $(2x\cos y + z\sin y)dx + (xz\cos y - x^2\sin y)dy + x\sin y\,dz$
$$= d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

Since x, y and z are independent variables,
$$\frac{\partial\phi}{\partial x} = 2x\cos y + z\sin y,\quad \frac{\partial\phi}{\partial y} = xz\cos y - x^2\sin y,\quad \frac{\partial\phi}{\partial z} = x\sin y$$

Integrating partially w.r.t. x, y & z respectively
$$\left.\begin{aligned} \phi &= x^2\cos y + xz\sin y + f(y,z) \\ \phi &= xz\sin y + x^2\cos y + g(x,z) \end{aligned}\right\} f, g \text{ \& } h \text{ are arbitrary functions}$$
$$\phi = xz\sin y + h(x,y)$$

Let us choose, $f(y,z) = c_1$, $g(x,z) = c_2$, $h(x,y) = x^2\cos y + c_3$.

$\therefore \phi = x^2\cos y + xz\sin y + c$; $c = c_1 + c_2 + c_3$, an arbitrary constant.

$\therefore$ The corresponding differential equation is given by: $(2x\cos y + z\sin y)dx + (xz\cos y - x^2\sin y)dy + x\sin y\,dz = 0$

$\Rightarrow d\phi = 0$, on integration, we have $\phi = $ constant.

i.e, the g.s. is given by $\boxed{x^2\cos y + xz\sin y = C}$

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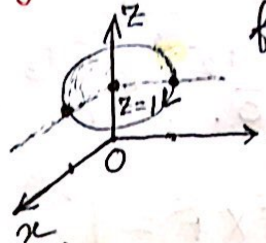
(51). Evaluate $\int_C \bar{A}\cdot d\bar{r}$ along the curve $x^2+y^2=1,\ z=1$ in the positive direction from $(0,1,1)$ to $(1,0,1)$ if $\bar{A}=(yz+2x)\hat{i}+xz\hat{j}+(xy+2z)\hat{k}$.

$x^2+y^2=1,\ z=1$

Let $x=\cos t,\ y=\sin t;\ z=1$.



from $(0,1,1)$ to $(1,0,1)$
i.e., $x: 0\to 1$
$y: 1\to 0$
then $t: \pi/2 \to 0$

Then $\bar{A}\cdot d\bar{r}$

$= (yz+2x)dx + xz\,dy + (xy+2z)dz$

$= (\sin t + 2\cos t)\, d(\cos t) + \cos t\, d(\sin t) + (\cos t \sin t + 2)\cdot d(1)$

$= [-\sin^2 t - \sin 2t + \cos^2 t]\,dt + 0$

$= (\cos 2t - \sin 2t)\,dt$

$\therefore \int \bar{A}\cdot d\bar{r} = \int_{t=\pi/2}^{0} (\cos 2t - \sin 2t)\,dt = \frac{1}{2}\left[\sin 2t + \cos 2t\right]_{t=\pi/2}^{0}$

$= \frac{1}{2}[1-\cos\pi] = \frac{1}{2}(1+1) = \underline{1}$. (Ans.)

(50) Show that the work done on a particle in moving it from A to B equals its change in kinetic energy at these points whether the force field is conservative or not.

Let us suppose a particle of constant mass m to move in this force field from A to B in space.

Now, $\bar{F} = m\bar{a} = m\frac{d^2\bar{r}}{dt^2}$.

Then $\bar{F}\cdot\frac{d\bar{r}}{dt} = m\frac{d^2\bar{r}}{dt^2}\cdot\frac{d\bar{r}}{dt} = \frac{m}{2}\frac{d}{dt}\left(\frac{d\bar{r}}{dt}\right)^2$.

Integrating w.r.t. t, we get

$$\int_A^B \bar{F}\cdot\frac{d\bar{r}}{dt}\,dt = \frac{m}{2}\int_A^B \frac{d}{dt}\left(\frac{d\bar{r}}{dt}\right)^2 dt$$

$\underline{\text{Work done}} = \int_A^B \bar{F}\cdot d\bar{r} = \left[\frac{m}{2}\left(\frac{d\bar{r}}{dt}\right)^2\right]_A^B = \frac{m}{2}v^2\Big]_A^B = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$

$= \underline{\text{change in K.E.}}$

Where v_A & v_B are the magnitudes of the velocities of the particle at A & B respectively.

~~If $\bar{F}$ be conservative, then $\bar{F} = -\nabla\phi$; ϕ is a scalar.~~

~~$\int_A^B \bar{F}\cdot d\bar{r} = -\int_A^B \nabla\phi\cdot d\bar{r} = -\int_A^B d\phi = \phi(A)-\phi(B)$~~

$\therefore$ The result follows whether F is conservative or not.